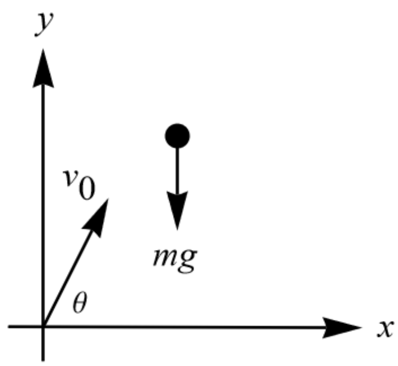
Problem 1.40

A cannon shoots a ball at an angle θ above the horizontal ground. **(a)** Neglecting air resistance, use Newton's second law to find the ball's position as a function of time. (Use axes with x measured horizontally and y vertically.) **(b)** Let $r(t)$ denote the ball's distance from the cannon. What is the largest possible value of θ if $r(t)$ is to increase throughout the ball's flight? [*Hint:* Using your solution to part (a) you can write down r^2 as $x^2 + y^2$, and then find the condition that r^2 is always increasing.]

Solution

Start by drawing a free-body diagram of the ball. Because there's no air resistance, there's only a gravitational force acting on the ball.



Newton's second law states that the sum of the forces on the ball is equal to its mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \\ \sum F_z = ma_z \end{cases}$$

The only force is due to gravity, and it's in the negative y-direction.

$$\begin{cases} 0 = ma_x \\ -mg = ma_y \\ 0 = ma_z \end{cases}$$

Divide both sides of each equation by m.

$$\begin{cases} 0 = a_x \\ -g = a_y \\ 0 = a_z \end{cases}$$

Acceleration is the second derivative of position.

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \dfrac{d^2y}{dt^2} = -g \\ \dfrac{d^2z}{dt^2} = 0 \end{cases}$$

Integrate both sides of each equation with respect to t to get the components of the ball's velocity.

$$\begin{cases} \dfrac{dx}{dt} = C_1 \\ \dfrac{dy}{dt} = -gt + C_2 \\ \dfrac{dz}{dt} = C_3 \end{cases} \tag{1}$$

The components of the initial velocity in the x-, y-, and z-directions are $v_\text{o}\cos\theta$, $v_\text{o}\sin\theta$, and 0, respectively.

$$\begin{aligned} \frac{dx}{dt}(0) &= C_1 = v_\text{o}\cos\theta & &\rightarrow & C_1 &= v_\text{o}\cos\theta \\ \frac{dy}{dt}(0) &= -g(0) + C_2 = v_\text{o}\sin\theta & &\rightarrow & C_2 &= v_\text{o}\sin\theta \\ \frac{dz}{dt}(0) &= C_3 = 0 & &\rightarrow & C_3 &= 0 \end{aligned}$$

As a result, equation (1) becomes

$$\begin{cases} \dfrac{dx}{dt} = v_\text{o}\cos\theta \\ \dfrac{dy}{dt} = -gt + v_\text{o}\sin\theta \\ \dfrac{dz}{dt} = 0 \end{cases}.$$

Integrate both sides of each equation with respect to t once more to get the components of the ball's position.

$$\begin{cases} x(t) = v_\text{o}t\cos\theta + C_4 \\ y(t) = -\dfrac{1}{2}gt^2 + v_\text{o}t\sin\theta + C_5 \\ z(t) = C_6 \end{cases} \tag{2}$$

The ball's initial position is the origin, so $x = 0$, $y = 0$, and $z = 0$ when $t = 0$.

$$\begin{aligned} x(0) &= v_\text{o}(0)\cos\theta + C_4 = 0 & &\rightarrow & C_4 &= 0 \\ y(0) &= -\frac{1}{2}g(0)^2 + v_\text{o}(0)\sin\theta + C_5 = 0 & &\rightarrow & C_5 &= 0 \\ z(0) &= C_6 = 0 & &\rightarrow & C_6 &= 0 \end{aligned}$$

Consequently, equation (2) becomes

$$\begin{cases} x(t) = v_o t\cos\theta \\ y(t) = -\dfrac{1}{2}gt^2 + v_o t\sin\theta \\ z(t) = 0 \end{cases}.$$

Therefore, the ball's position is

$$\boxed{\mathbf{r}(t) = \left\langle v_o t\cos\theta, -\frac{1}{2}gt^2 + v_o t\sin\theta, 0\right\rangle}.$$

Notice that this formula can be obtained by taking the limit of the position vector in Problem 1.39 as $\phi \to 0$. The distance of the ball from the origin is

$$\begin{aligned} r(t) &= |\mathbf{r}(t)| \\ &= \sqrt{(v_o t\cos\theta)^2 + \left(-\frac{1}{2}gt^2 + v_o t\sin\theta\right)^2 + 0^2} \\ &= \sqrt{v_o^2 t^2\cos^2\theta + \frac{1}{4}g^2t^4 - gt^3 v_o\sin\theta + v_o^2 t^2\sin^2\theta} \\ &= \sqrt{\frac{1}{4}g^2t^4 - gt^3 v_o\sin\theta + v_o^2 t^2} \end{aligned}$$

For the distance from the origin to the ball to always be increasing, the first derivative of $r(t)$ must be greater than or equal to zero for all time. $r(t)$ is a square root function, which is monotonically increasing, so $[r(t)]^2$ will have the same behavior. $[r(t)]^2$ is preferable to work with because it has no square root.

$$\begin{aligned} \frac{d}{dt}[r(t)] \geq 0 \quad \Rightarrow \quad & \frac{d}{dt}[r(t)]^2 \geq 0 \\ & \frac{d}{dt}\left(\frac{1}{4}g^2t^4 - gt^3 v_o\sin\theta + v_o^2 t^2\right) \geq 0 \\ & g^2t^3 - 3gt^2 v_o\sin\theta + 2v_o^2 t \geq 0 \\ & t(g^2t^2 - 3gt v_o\sin\theta + 2v_o^2) \geq 0 \end{aligned}$$

Since $t \geq 0$, the quadratic expression in parentheses must be greater than or equal to zero.

$$g^2t^2 - 3gt v_o\sin\theta + 2v_o^2 \geq 0$$

This inequality describes a parabola opening upward that has at most one zero; therefore, the discriminant is not positive.

$$(-3gv_o\sin\theta)^2 - 4(g^2)(2v_o^2) \leq 0$$

Solve for θ.

$$9g^2v_o^2\sin^2\theta - 8g^2v_o^2 \leq 0$$

$$9\sin^2\theta - 8 \leq 0$$

$$\sin^2\theta \leq \frac{8}{9}$$

$$-\sqrt{\frac{8}{9}} \leq \sin\theta \leq \sqrt{\frac{8}{9}}$$

Since $\theta > 0$, only positive values of $\sin\theta$ need to be considered.

$$0 < \sin\theta \leq \sqrt{\frac{8}{9}}$$

The largest possible value of θ if $r(t)$ is to increase throughout the ball's flight is then

$$\theta_{\max} = \sin^{-1}\left(\sqrt{\frac{8}{9}}\right) \approx 1.23 \text{ radians}$$

$$\approx 70.5^\circ.$$